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Bases of Bethe Vectors and Difference Equations with Regular Singular Points

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Abstract. We prove that Bethe vectors generically form a base in a tensor product of irreducible highest weight $\mathfrak{sl}_2$-modules or $U_q(\mathfrak{sl}_2)$-modules. We apply this result to difference equations with regular singular points. We show that if such an equation has local solutions at each of its singular point, then generically it has a polynomial solution.

Introduction

The Bethe ansatz is a large collection of methods in the theory of quantum integrable models to calculate the spectrum and eigenvectors for a certain commutative subalgebra of the algebra of observables for an integrable model. This commutative subalgebra includes the Hamiltonian of the model. Its elements are called integrals of motion or conservation laws of the model. The most part of the recent development of the Bethe ansatz methods is due to the quantum inverse scattering transform invented by the Leningrad-St. Petersburg school of mathematical physics. The bibliography on the Bethe ansatz is enormous. We refer a reader to reviews [BIK], [F], [FT].

Usually in the framework of the Bethe ansatz eigenvalues of conservation laws are expressed as functions of some additional parameters which have to obey a system of nonlinear equations. This system is called the system of Bethe ansatz equations. In the algebraic Bethe ansatz there is also a remarkable vector-valued function of the same additional variables. Its values at solutions to the system of Bethe ansatz equations are common eigenvectors of conservation laws. These common eigenvectors are called the Bethe vectors. An important problem is to show that the number of appropiate solutions to the system of Bethe ansatz equations is equal to the dimension of the representation space of the algebra of observables and the corresponding Bethe vectors form a base of this space.

In this paper we solve this problem for generic integrable models associated to a finite tensor product of irreducible highest weight $\mathfrak{sl}_2$-modules or $U_q(\mathfrak{sl}_2)$-modules. A similar but weaker result was announced in a recent preprint [LS], see the remark after Theorem 4.2.

A traditional method to study the system of Bethe ansatz equations is based on the so-called string hypothesis which predicts a certain behaviour of solutions. The string hypothesis was used to get many important results in physical models. Partially, these results were verified by other methods.

The string hypothesis motivated deep combinatorial results, see [Ki1], [Ki2]. In particular, it was shown that the string hypothesis predicts the correct number of appropriate solutions to the system of Bethe ansatz equations. This fact is called the combinatorial completeness of the Bethe vectors.

A disadvantage of the string hypothesis is that it is, strictly speaking, false and fails to predict even a qualitative picture of solutions to the Bethe ansatz equations [EKS].

The string hypothesis needs to be understood better. We expect that some clarification of the string hypothesis could come from the analysis of the quantized Knizhnik-Zamolodchikov (qKZ) equations. In [TV] we showed that Bethe vectors are the first terms of asymptotic solutions to the qKZ equations.

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One could expect that there is a way to count solutions to the qKZ equations (not the Bethe vectors) which is similar to the counting of the Bethe vectors in the string hypothesis.

The system of Bethe ansatz equations has intimate relation to a certain class of difference equations which can be called difference equations with regular singular points. These difference equations arise in two-dimensional exactly solvable models in statistical mechanics [B] as well as in separation of variables in quantum lattice integrable models [S2], [S3]. We apply our results on Bethe vectors to these difference equations to show that such a difference equation has a polynomial solution if it has local solutions at singular points. We also count the number of difference equations having polynomial solutions.

The paper is organized as follows. In the first section we study general case related to a tensor product of irreducible highest weight $\mathfrak{sl}_2$-modules. In the second section we consider the special case, where the Bethe vectors are singular vectors for the standard $\mathfrak{sl}_2$ action in a tensor product of $\mathfrak{sl}_2$-modules. The third section contains the application to difference equations with regular singular points. Results in the $U_q(\mathfrak{sl}_2)$ case are given in the fourth section for generic q and in the fifth section for q being a root of unity. In the last section we consider multiplicative difference equations with regular singular points.

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1. Bases of Bethe vectors in $\mathfrak{sl}_2$-modules

Consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$ with generators e, f, h:

$$[h, e] = e\,, \qquad [h, f] = -f\,, \qquad [e, f] = 2h\,.$$

Let $M = \mathrm{End}\,(\mathbb{C}^2)$. Introduce $T(u) \in M[u] \otimes U(\mathfrak{g})$ as follows:

$$T(u) = \begin{pmatrix} u+h & f \\ e & u-h \end{pmatrix}.$$

Let ι_m be the embedding $U(\mathfrak{g}) \to U(\mathfrak{g})^{\otimes n}$ as the m-th tensor factor. Let $\delta : U(\mathfrak{g}) \to U(\mathfrak{g})^{\otimes n}$ be the canonical embedding which coincide with $\iota_1 + \ldots + \iota_n$ on $\mathfrak{g}$.

Let $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. Let $T_m(u) = \mathrm{id} \otimes \iota_m\bigl(T(u)\bigr) \in M[u] \otimes U(\mathfrak{g})^{\otimes n}$. Set

$$\mathrm{T}(u) = T_1(u - z_1) \ldots T_n(u - z_n) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

where $A(u)$, $B(u)$, $C(u)$ and $D(u)$ are suitable elements in $\mathbb{C}\,[u] \otimes U(\mathfrak{g})^{\otimes n}$.

Let $\kappa \in \mathbb{C}$. Set $\mathcal{T}(u) = A(u) + \kappa D(u)$. Coefficients of the polynomial $\mathcal{T}(u)$ generate a remarkable commutative subalgebra in $U(\mathfrak{g})^{\otimes n}$. The case $\kappa = 1$ is of special interest because $\mathcal{T}(u)$ commute with $\delta\bigl(U(\mathfrak{g})\bigr)$ in this case.

Remark. The construction explained above originates from the theory of quantum integrable models. $\mathcal{T}(u)$ is called there the *transfer-matrix*. Its coefficients are conservation laws (integrals of motion) of the correspoding lattice model. For more detailed explanation of the quantum inverse scattering transform and the algebraic Bethe ansatz as well as for the related bibliography we refer a reader to [BIK], [F], [FT].

Let $V_1, \ldots, V_n$ be irreducible highest weight $\mathfrak{g}$-modules with highest weights $\Lambda_1, \ldots, \Lambda_n$, respectively. Set $V = V_1 \otimes \ldots \otimes V_n$. A problem of the theory of quantum integrable models is to diagonalize the operators $\mathcal{T}(u)$ in the space V. The *algebraic Bethe ansatz* is a tool to construct eigenvectors and eigenvalues of $\mathcal{T}(u)$. One considers a special V-valued function $w(t_1, \ldots, t_\ell)$ of auxiliary variables $t = (t_1, \ldots, t_\ell)$ and chooses t in such a way that $w(t)$ becomes an eigenvector. The arising conditions on t are called the Bethe ansatz equations and the corresponding eigenvector is called a Bethe vector. In this paper we prove that for generic $z_1, \ldots, z_n$ and κ the Bethe vectors form a base in V.

More precisely, let $v_1, \ldots, v_n$ be generating vectors of $\mathfrak{g}$-modules $V_1, \ldots, V_n$, respectively. Let $\ell \in \mathbb{Z}_{\geqslant 0}$. Let $V_{[\ell]} \subset V$ be a weight subspace: $V_{[\ell]} = \{\, v \in V \mid \delta(h) \cdot v = \bigl(\sum_{m=1}^{n} \Lambda_m - \ell\bigr)\, v \,\}$. Let $t = (t_1, \ldots, t_\ell) \in \mathbb{C}^{\,\ell}$. Set

$$w(t) = B(t_1) \ldots B(t_\ell) \cdot v_1 \otimes \ldots \otimes v_n\,.$$

$w(t)$ is a $V_{[\ell]}$-valued symmetric polynomial in variables $t_1, \dots, t_\ell$. Another formula for $w(t)$ see below in Lemma 1.12.

For given $z_1, \dots, z_n$ consider a system of algebraic equations on variables $t_1, \dots, t_\ell$:

$$\prod_{m=1}^{n} (t_a - z_m + \Lambda_m) \prod_{\substack{b=1 \\ b \neq a}}^{\ell} (t_a - t_b - 1) = \kappa \prod_{m=1}^{n} (t_a - z_m - \Lambda_m) \prod_{\substack{b=1 \\ b \neq a}}^{\ell} (t_a - t_b + 1) . \tag{1.1}$$

$a = 1, \dots, \ell$. This system is called the system of *Bethe ansatz equations*. A solution t to system (1.1) is called *offdiagonal* if $t_1, \dots, t_\ell$ are pairwise distinct, and *diagonal* otherwise.

Set

$$\tau(u,t) = \prod_{m=1}^{n} (u - z_m + \Lambda_m) \prod_{a=1}^{\ell} \frac{u - t_a - 1}{u - t_a} + \kappa \prod_{m=1}^{n} (u - z_m - \Lambda_m) \prod_{a=1}^{\ell} \frac{u - t_a + 1}{u - t_a} . \tag{1.2}$$

(1.1) Theorem. [BIK], [F], [FT] *Let $t_1, \dots, t_\ell$ be an offdiagonal solution to system* (1.1). *Then* $\mathcal{T}(u) \cdot w(t) = \tau(u,t)\, w(t)$.

Remark. Let $t \in \mathbb{C}^\ell$ be such that its coordinates are pairwise distinct. Then t is a solution to system (1.1) iff $\tau(u,t)$ is a polynomial in u.

Define a set $\mathfrak{Z}$ in $\mathbb{C}^\ell$ by the equation

$$\prod_{a=1}^{\ell} \Big(\prod_{m=1}^{n} (t_a - z_m + \Lambda_m)(t_a - z_m - \Lambda_m) \prod_{\substack{b=1 \\ b \neq a}}^{\ell} (t_a - t_b - 1) \Big) = 0 .$$

A solution t to system (1.1) is called *admissible* if $t \notin \mathfrak{Z}$ and *unadmissible* otherwise. For an admissible offdiagonal solution t the vector $w(t)$ is called the *Bethe vector*. A solution t to system (1.1) is called a *trivial* solution if $w(t) = 0$ and *nontrivial* otherwise.

System (1.1) is preserved by the natural action of the symmetric group $\mathbf{S}_\ell$ on variables $t_1, \dots, t_\ell$. Therefore, $\mathbf{S}_\ell$ acts on solutions to this system. Let $\mathfrak{C}$ be the set of $\mathbf{S}_\ell$-orbits of admissible offdiagonal solutions.

Say that $z_1, \dots, z_n$ are *well separated* if all points $z_m - \Lambda_m + s$, $s \in \mathbb{Z}_{\geqslant 0}$, $s < 2\Lambda_m$ for $2\Lambda_m \in \mathbb{Z}_{\geqslant 0}$, and $z_m + \Lambda_m$ $m = 1, \dots, n$, are pairwise distinct.

Remark. $z_1, \dots, z_n$ are well separated if and only if the module V enjoys the next properties:
 i) V is irreducible with respect to the subalgebra generated by coefficients of polynomials $A(u)$, $B(u)$, $C(u)$, $D(u)$.
 ii) The commutative subalgebra generated by coefficients of $A(u)$ acts in V in a semisimple way.

This follows from results of [T], [NT]. The part "*only if*" also follows from results of the present paper. Note in addition that the action in V of the subalgebra generated by coefficients of $A(u)$ has simple spectrum.

(1.2) Theorem. *Let κ be generic. Let $z_1, \dots, z_n$ be well separated. Then*
 a) *All admissible offdiagonal solutions to system* (1.1) *are nondegenerate.*
 b) *All unadmissible offdiagonal solutions to system* (1.1) *are trivial.*
 c) $\#\mathfrak{C} = \dim V_{[\ell]}$ *and the corresponding Bethe vectors form a base in* $V_{[\ell]}$.

Remark. This Theorem was proved in [TV] for generic $z_1, \dots, z_n$, $\Lambda_1, \dots, \Lambda_n$. The proof of Theorem 1.2 is similar to the corresponding proof in [TV]. To make this paper self-contained we reproduce some lemmas from [TV].

To prove Theorem 1.2 we use the following strategy. Consider the limit of system (1.1) as $\kappa \to 0$:

$$\prod_{m=1}^{n} (t_a - z_m + \Lambda_m) \prod_{\substack{b=1 \\ b \neq a}}^{\ell} (t_a - t_b - 1) = 0 , \qquad a = 1, \dots, \ell . \tag{1.3}$$

All solutions to this system are isolated. We consider their deformations for $\kappa \neq 0$. We show that for generic $z_1, \dots, z_n$ and κ, offdiagonal solutions to system (1.1) are nondegenerate and they are deformations of offdiagonal solutions to system (1.3). The coordinates of an offdiagonal solution to system (1.3) form a union of arithmetic progressions starting at points $z_1, \dots, z_n$. The corresponding solution to system (1.1) is admissible iff each of the arithmetic progression is not too long. This proves the first part of claim c). To prove the second part of claim c) we compute explicitly limits of Bethe vectors as $\kappa \to 0$ and show that they form a base in $V_{[\ell]}$. The claim b) follows from Theorem 1.13.

(1.3) Lemma. *Let* $2\Lambda_m \notin \mathbb{Z}_{\geqslant 0}$, $m = 1, \ldots, n$. *For* $\kappa \neq 0$ *and well separated* $z_1, \ldots, z_n$ *all solutions to system* (1.1) *are admissible.*

Proof. The lemma easily follows from direct analysis of system (1.1). As an example consider the case $\ell = 2$. Take a solution $t \in \mathfrak{Z}$. Suppose $t_1 = z_m - \Lambda_m$. Then from the first equation, $t_2 = t_1 + 1$ and the second equation cannot be satisfied. If $t_1 = t_2 + 1$, then from the first equation, $t_1 = z_m + \Lambda_m$ for some m, and the second equation cannot be satisfied. Similarly, we can start from $t_1 = z_m + \Lambda_m$ or $t_1 = t_2 - 1$. All the other cases can be obtained by the action of the symmetric group $\mathbf{S}_2$. $\square$

(1.4) Lemma. *Let* $\Lambda = \sum_{m=1}^{n} \Lambda_m$. *Assume that at least one of the following conditions is fulfilled:* a) $\kappa \neq 1$; b) $2\Lambda \notin \mathbb{Z}_{\geqslant 0}$; c) $2\Lambda < \ell - 1$; d) $\Lambda > \ell - 1$. *Then all offdiagonal solutions to system* (1.1) *are isolated.*

Proof. Assume that there is a nonisolated offdiagonal solution to system (1.1). This means that we have a curve $t(s)$, $s \in \mathbb{R}$, such that $t(s)$ is an offdiagonal solution to system (1.1) for any s. Moreover, we can assume that as $s \to +\infty$, $t(s)$ tends to infinity in the following way: $t_a(s) \to \infty$ if $a > f$ and $t_a(s)$ has a finite limit if $a \leqslant f$. Set

$$\tau_0(u) = \prod_{m=1}^{n} (u - z_m + \Lambda_m) \prod_{a=1}^{f} \frac{u - t_a(+\infty) - 1}{u - t_a(+\infty)} + \kappa \prod_{m=1}^{n} (u - z_m - \Lambda_m) \prod_{a=1}^{f} \frac{u - t_a(+\infty) + 1}{u - t_a(+\infty)} . \tag{1.4}$$

It is clear that for any u such that $u \neq t_a(+\infty)$, $a = 1, \ldots, f$,

$$\tau(u, t(s)) \to \tau_0(u) \tag{1.5}$$

as $\kappa \to \kappa_0$. Since $t(s)$ is an offdiagonal solution to system (1.1), $\tau(u, t(s))$ is a polynomial in u for any s and $\tau_0(u)$ is a polynomial in u as well. Hence, relation (1.5) is valid for any u and coefficients of $\tau(u, t(s))$ tend to coefficients of $\tau_0(u)$ as $s \to +\infty$. From (1.2) and (1.4) we have that

$$\tau(u, t(s)) - \tau_0(u) = (1 - \kappa)(f - \ell)\, u^{n-1} + \ldots .$$

This means that $\kappa = 1$. Similarly, for $\kappa = 1$ we have

$$\tau(u, t(s)) - \tau_0(u) = \bigl(f(f - 2\Lambda - 1) - \ell(\ell - 2\Lambda - 1)\bigr)\, u^{n-2} + \ldots .$$

Since $0 \leqslant f < \ell$, this means that $f = 2\Lambda + 1 - \ell$, $2\Lambda \in \mathbb{Z}_{\geqslant 0}$ and $\ell - 1 \leqslant 2\Lambda \leqslant 2\ell - 2$. The lemma is proved. $\square$

(1.5) Lemma. *Consider solutions to system* (1.1) *as (multivalued) functions of* κ. *For any offdiagonal solution to system* (1.1) *every its branch remains finite for any* $\kappa \neq 1$.

Proof. Let $t(\kappa)$ be an offdiagonal solution to system (1.1). Suppose, $t(\kappa)$ tends to infinity, as $\kappa \to \kappa_0 \neq 1$. We can assume that $t_a(\kappa) \to \infty$ if $a > f$ and $t_a(\kappa)$ has a finite limit if $a \leqslant f$. Set

$$\tau_0(u) = \prod_{m=1}^{n} (u - z_m + \Lambda_m) \prod_{a=1}^{f} \frac{u - t_a(\kappa_0) - 1}{u - t_a(\kappa_0)} + \kappa_0 \prod_{m=1}^{n} (u - z_m - \Lambda_m) \prod_{a=1}^{f} \frac{u - t_a(\kappa_0) + 1}{u - t_a(\kappa_0)} . \tag{1.6}$$

Since $t(\kappa)$ is an offdiagonal solution to system (1.1), $\tau(u, t(\kappa))$ is a polynomial in u for any $\kappa \neq \kappa_0$ and $\tau_0(u)$ is a polynomial in u as well. Similar to the proof of Lemma 1.4 we have that coefficients of $\tau(u, t(\kappa))$ tend to coefficients of $\tau_0(u)$ as $\kappa \to \kappa_0$. On the other hand, it follows from (1.2) and (1.6) that

$$\tau(u, t(\kappa)) - \tau_0(u) = (\kappa - \kappa_0)\bigl(u^n + \bigl(\ell - \sum_{m=1}^{n} (z_m + \Lambda_m)\bigr) u^{n-1}\bigr) + (1 - \kappa_0)(f - \ell)\, u^{n-1} + \ldots$$

which implies $f = \ell$. The lemma is proved. $\square$

Set $\mathbb{O} = \{ \eta \in \mathbb{Z}_{\geqslant 0}^{n\ell} \mid \sum_{m=1}^{n} \sum_{a=1}^{\ell} \eta_{ma} = \ell$ and $\eta_{ma} = 0 \Rightarrow \eta_{m,a+1} = 0 \quad m = 1, \ldots, n, \ a = 1, \ldots, \ell \}$. Let $\mathcal{O}(\eta)$ be an $\mathbf{S}_\ell$-orbit of solutions to system (1.3) fixed by conditions

$$\#\{b \mid t_b = z_m - \Lambda_m + a - 1\} = \eta_{ma} , \qquad m = 1, \ldots, n, \quad a = 1, \ldots, \ell .$$

(1.6) Lemma.

a) *All solutions to system* (1.3) *are isolated.*

b) *Any* $\mathbf{S}_\ell$*-orbit of solution to system* (1.3) *has the form* $\mathcal{O}(\eta)$ *for a suitable* $\eta \in \mathbb{O}$.

c) *For generic* $z_1, \ldots, z_n$ $\mathbf{S}_\ell$*-orbits* $\mathcal{O}(\eta)$ *of solutions to system* (1.3) *are pairwise distinct.*

d) *For generic* $z_1, \ldots, z_n$ *all offdiagonal solutions to system* (1.3) *are nondegenerate.*

Proof. Claim a) follows from the next lemma.

(1.7) Lemma. *Let* $Q_1, \ldots, Q_l$ *be polynomials in one variable,* $\deg Q_a > 0$, $a = 1, \ldots, l$. *Then all solutions to the system*

$$Q_a(t_a) \prod_{\substack{b=1 \\ b \neq a}}^{l} (t_a - t_b - 1) = 0\,, \qquad a = 1, \ldots, l\,,$$

for variables $t_1, \ldots, t_l$ *are isolated.*

Proof. We prove the lemma by induction on l. The case $l = 1$ is clear.

Let $t_1, \ldots, t_l$ be a solution to the system in question. Without loss of generality we can assume that $\operatorname{Re} t_1 = \ldots = \operatorname{Re} t_f < \operatorname{Re} t_{f+1} \leqslant \ldots \leqslant \operatorname{Re} t_l$. Therefore, we have that $Q_a(t_a) = 0$, $a = 1, \ldots, f$. Remaining variables $t_{f+1}, \ldots, t_l$ satisfies a new system

$$\tilde{Q}_a(t_a) \prod_{\substack{b=f+1 \\ b \neq a}}^{l} (t_a - t_b - 1) = 0\,, \qquad a = f+1, \ldots, l\,, \tag{1.7}$$

where

$$\tilde{Q}_a(t_a) = Q_a(t_a) \prod_{b=1}^{f} (t_a - t_b - 1)\,.$$

By the induction assumption all solution to system (1.7) are isolated. Since possible values of $(t_1, \ldots, t_f)$ form a discrete set, the lemma is proved. □

It is clear that the proof of Lemma 1.7 implies claim b) as well. Claim c) is evident. To prove claim d) we drop all factors which remain nonzero on the solution in question and the apply the next lemma.

(1.8) Lemma. *Let* $Q_1, \ldots, Q_l$ *be homogeneous polynomials in variables* $x_1, \ldots, x_l$. *Assume, that* $x_a = 0$, $a = 1, \ldots, l$, *is an isolated solution to a system*

$$Q_a(x) = 0\,, \qquad a = 1, \ldots, l\,.$$

Then the multiplicity of this solution is equal to $\prod_{a=1}^{l} \deg Q_a$.

Lemma 1.7 is proved. □

Remark. Lemma 1.8 corresponds to Lemma 4.9 in [TV]. The form in which Lemma 4.9 is formulated in [TV] is wrong. References to Lemma 4.9 in [TV] must be replaced by references to the above Lemma 1.8. All results in [TV] remain correct.

(1.9) Lemma. *Let* $z_1, \ldots, z_n$ *be generic. Let* $t(\kappa)$ *be a solution to system* (1.1), *which is a deformation of a diagonal solution* $t(0)$ *to system* (1.3). *Then* $t(\kappa)$ *is a diagonal solution.*

Proof. Let $a_0 = \min\{\, a \mid \eta_{ma} > 1 \text{ for some } m \,\}$. Let m_0 be such that $\eta_{m_0 a_0} > 1$. Let, for example, $t_1 = t_\ell = z_{m_0} + a_0 - 1$. Let $t^- = (t_1, \ldots, t_{\ell-1}) \in \mathbb{C}^{\ell-1}$. Consider a new system

$$\prod_{m=1}^{n} (t_1 - z_m + \Lambda_m) \prod_{b=2}^{\ell-1} (t_1 - t_b - 1) = -\kappa \prod_{m=1}^{n} (t_1 - z_m - \Lambda_m) \prod_{b=2}^{\ell-1} (t_1 - t_b + 1)\,, \tag{1.8}$$

$$(t_a - t_1 - 1) \prod_{m=1}^{n} (t_a - z_m + \Lambda_m) \prod_{\substack{b=1 \\ b \neq a}}^{\ell-1} (t_a - t_b - 1) = \kappa\, (t_a - t_1 + 1) \prod_{m=1}^{n} (t_a - z_m - \Lambda_m) \prod_{\substack{b=1 \\ b \neq a}}^{\ell-1} (t_a - t_b + 1)\,,$$

$a = 2, \ldots, \ell - 1$. It is obtained from system (1.1) by the substitution $t_\ell = t_1$. Incidently, the first and the last equations of (1.1) coincide after this substitution. Any solution to system (1.8) gives rise to a diagonal solution to system (1.1) by setting $t_\ell = t_1$.

By Lemma 1.7 $t^-(0)$ is an isolated solution to system (1.8) at $\kappa = 0$. It follows from Lemma 1.8 that the solution $t(0)$ to system (1.3) has the same multiplicity as the solution $t^-(0)$ to system (1.8) at $\kappa = 0$. Namely, the prescribed choice of coinciding coordinates of the solution $t(0)$ to system (1.3) guarantees that a homogeneous polynomial corresponding to the dropped equation with $a = \ell$ is of degree 1. This choice also guarantees that degrees of homogeneous polynomials corresponding to the remaining equations with $a = 1, \ldots, \ell - 1$ are the same for both systems (1.3) and (1.8) at $\kappa = 0$.

The coincidence of multiplicities means that any deformation of the diagonal solution $t(0)$ can be obtained from some deformation of the solution $t^-(0)$ and, therefore, is diagonal. □

Proof of Theorem 1.2. Henceforward, until the end of the section we assume that κ is generic, unless the contrary is indicated explicitly. Let $z_1, \ldots, z_n$ be generic, until the contrary is indicated explicitly.

Set $\mathcal{Z}_\ell = \{\, \nu \in \mathbb{Z}^n_{\geqslant 0} \mid \sum_{m=1}^{n} \nu_m = \ell \,\}$. Set $\mathcal{Z}^\circ_\ell = \{\, \nu \in \mathbb{Z}^n_{\geqslant 0} \mid \sum_{m=1}^{n} \nu_m = \ell$ and $\nu_m \leqslant 2\Lambda_m$, if $2\Lambda_m \in \mathbb{Z}_{\geqslant 0}$, $m = 1, \ldots, n \,\}$. For $\nu \in \mathcal{Z}_\ell$ let $t^\star(\nu)$ be the following offdiagonal solution to system (1.3)

$$t^\star_a(\nu) = z_m - \Lambda_m + a - \ell_{m-1} - 1\,, \qquad \ell_{m-1} < a \leqslant \ell_m\,, \tag{1.9}$$

where $\ell_m = \sum_{k=1}^{m} \nu_k$, $\ell_0 = 0$, $\ell_n = \ell$. It is related to the previous description of solutions as $\eta_{ka} = 1$ for $a \leqslant \nu_k$ and $\eta_{ka} = 0$ for $a > \nu_k$.

Any $\mathbf{S}_\ell$-orbit of offdiagonal solutions to system (1.3) can be obtained as the orbit of a solution $t^\star(\nu)$ for a suitable ν. Moreover, for generic $z_1, \ldots, z_n$ solutions $t^\star(\nu)$ for different ν belong to different $\mathbf{S}_\ell$-orbits. Let $t(\nu, \kappa)$ be a solution to system (1.1) which is the deformation of $t^\star(\nu)$.

(1.10) Lemma. *Let $\nu \in \mathcal{Z}^\circ_\ell$. Then $t(\nu, \kappa)$ is an admissible solution.*

Proof. By direct analysis of system (1.1) we obtain that

$$t_a(\nu, \kappa) = t^\star(\nu) + \kappa^{\ell_i - a + 1} t'_a(\nu, \kappa)\,, \qquad t'_a(\nu, 0) \neq 0\,,$$

for $\ell_{i-1} < a \leqslant \ell_i$. This means that $t(\nu, \kappa)$ is an admissible solution for small κ, and hence, for generic κ. □

(1.11) Lemma. *Let $\nu \in \mathcal{Z}_\ell \setminus \mathcal{Z}^\circ_\ell$. Then $t(\nu, \kappa)$ is an unadmissible solution.*

Proof. Let, for example, $\nu_1 > 2\Lambda_1$. Set $d = 2\Lambda_1 + 1$. Let $t^- = (t_{d+1}, \ldots, t_\ell) \in \mathbb{C}^{\ell - d}$. Consider a system for $t_{d+1}, \ldots, t_\ell$

$$\prod_{m=1}^{n} \bigl(t_a - z_m + \Lambda_m\bigr) \prod_{\substack{b=1 \\ b \neq a}}^{\ell} (t_a - t_b - 1) = \kappa \prod_{m=1}^{n} \bigl(t_a - z_m - \Lambda_m\bigr) \prod_{\substack{b=1 \\ b \neq a}}^{\ell} (t_a - t_b + 1)\,, \tag{1.10}$$

$a = d + 1, \ldots, \ell$ where $t_c = t^\star_c(\nu)$ for $c = 1, \ldots, d$. This system is obtained from system (1.1) by the substitution $t_c = t^\star_c(\nu)$, $c = 1, \ldots, d$. Incidently, the first d equations of (1.1) become identities after this substitution. Any solution to system (1.10) gives rise to an unadmissible solution to system (1.1) by setting $t_c = t^\star_c(\nu)$, $c = 1, \ldots, d$. The multiplicity of solution $t^\star(\nu)$ to system (1.3) is equal to 1 since $t^\star(\nu)$ is offdiagonal. This means that the multiplicity of the solution $t^-(\nu, 0)$ to system (1.10) at $\kappa = 0$ is also equal to 1. Moreover, the solution $\tilde{t}(\nu, \kappa)$ to system (1.1) can be obtained from the solution $t^-(\nu, \kappa)$ to system (1.10) and, therefore, is unadmissible. □

Finally, there are precisely $\#\mathcal{Z}^\circ_\ell = \dim V_{[\ell]}$ $\mathbf{S}_\ell$-orbits of admissible offdiagonal solutions. All of them are the orbits of solutions $t(\nu, \kappa)$ for a suitable $\nu \in \mathcal{Z}^\circ_\ell$ and, hence, are nondegenerate.

Introduce the canonical monomial base in V: $\{\, F^\nu = f^{\nu_1} v_1 \otimes \ldots \otimes f^{\nu_n} v_n \,\}$. It is clear that $\{\, F^\nu \mid \nu \in \mathcal{Z}^\circ_\ell \,\}$ is a base in $V_{[\ell]}$.

(1.12) Lemma. [Ko] *The following decomposition holds*

$$w(t)=\sum_{part}F^{\nu(\Gamma)}\prod_{l=2}^{n}\prod_{m=1}^{l-1}\Bigl(\prod_{\substack{a\in\Gamma_l\\ b\in\Gamma_m}}\frac{t_a-t_b-1}{t_a-t_b}\prod_{a\in\Gamma_l}(t_a-z_m+\Lambda_m)\prod_{a\in\Gamma_m}(t_a-z_l-\Lambda_l)\Bigr).$$

Here the sum is taken over all partitions of the set $\{1,\dots,\ell\}$ *into disjoint subsets* $\Gamma_1,\dots,\Gamma_n$ *and* $\nu(\Gamma)=(\#\Gamma_1,\dots,\#\Gamma_n)$.

Define on $\mathcal{Z}_\ell$ a lexicographical order: $\nu<\nu'$ if $\nu_1<\nu_1'$, or $\nu_1=\nu_1'$, $\nu_2<\nu_2'$, etc. Say that $b\ll a$ if $b\leqslant\ell_m<a$ for some m. Lemma 1.12 implies that for $t=t^*(\nu)$

$$w(t)=F^{\nu}\prod_{a=2}^{\ell}\prod_{b\ll a}\frac{t_a-t_b-1}{t_a-t_b}\prod_{m=1}^{n}\Bigl(\prod_{a>\ell_m}(t_a-z_m+\Lambda_m)\prod_{a\leqslant\ell_{m-1}}(t_a-z_m-\Lambda_m)\Bigr)+\sum_{\nu'>\nu}F^{\nu'}\theta_{\nu\nu'}\,.$$

where $\theta_{\nu\nu'}$ are suitable coefficients. This means that $\{\,w\bigl(t(\nu,\kappa)\bigr)\mid\nu\in\mathcal{Z}_\ell^\circ\,\}$ is a base in $V_{[\ell]}$. Hence, $\{\,w\bigl(t(\nu,\kappa)\bigr)\mid\nu\in\mathcal{Z}_\ell^\circ\,\}$ is a base in $V_{[\ell]}$ for generic κ.

Let v_m^* be a linear function on V_m such that $\langle v_m^*,v_m\rangle=1$ and $\langle v_m^*,v\rangle=0$ for any weight vector $v\in V_m$, $v\neq v_m$.

(1.13) Theorem. [Ko], [TV] *Let* $\Lambda_1,\dots,\Lambda_n$, $z_1,\dots,z_n$ *and* κ *be generic.*
a) *For any offdiagonal solution* $t=(t_1,\dots,t_n)$ *to system* (1.1)

$$\langle v_1^*\otimes\ldots\otimes v_n^*,C(t_1)\ldots C(t_n)w(t)\rangle=(-1)^{\ell}\prod_{m=1}^{n}\prod_{a=1}^{\ell}(t_a-z_m)-\Lambda_m)\prod_{a=2}^{\ell}\prod_{b=1}^{a-1}\frac{t_a-t_b+1}{t_a-t_b}\times$$
$$\times\det\Bigl[\frac{\partial}{\partial t_a}\Bigl(\prod_{m=1}^{n}(t_b-z_m+\Lambda_m)\prod_{\substack{c=1\\ c\neq b}}^{\ell}(t_b-t_c-1)-\kappa\prod_{m=1}^{n}(t_b-z_m-\Lambda_m)\prod_{\substack{c=1\\ c\neq b}}^{\ell}(t_b-t_c+1)\Bigr)\Bigr]_{a,b=1,\dots,\ell}.$$

b) *For any offdiagonal solutions* t *and* $\tilde t=(\tilde t_1,\dots,\tilde t_n)$ *to system* (1.1) *which lie in different* $\mathbf{S}_\ell$*-orbits*

$$\langle v_1^*\otimes\ldots\otimes v_n^*,C(\tilde t_1)\ldots C(\tilde t_n)w(t)\rangle=0\,.$$

c) *Let* $t,\tilde t$ *be isolated solutions to system* (1.1). *Then both claims* a) *and* b) *remain valid for any* Λ_1, $\dots,\Lambda_n$, $z_1,\dots,z_n$ *and* κ.

Remark. In this paper we use a normalization of $w(t)$ which differs from the normalization in [TV].

Let $\tilde t$ be an unadmissible offdiagonal solution to system (1.1). Since $\{\,w\bigl(t(\nu,\kappa)\bigr)\mid\nu\in\mathcal{Z}_\ell^\circ\,\}$ is a base in $V_{[\ell]}$ for generic κ and $z_1,\dots,z_n$, Theorem 1.13 implies $w(\tilde t)=0$. The theorem is proved for generic $z_1,\dots,z_n$.

Let $z_1,\dots,z_n$ be well separated, but not necessarily generic.

(1.14) Lemma. *Let* κ *be generic. Then* $t(\nu,\kappa)$, $\nu\in\mathcal{Z}_\ell^\circ$, *are nondegenerate admissible offdiagonal solutions to system* (1.1) *and the corresponding Bethe vectors form a base in* $V_{[\ell]}$.

The proof is the same as for generic $z_1,\dots,z_n$.

Since by Lemma 1.4 all offdiagonal solutions to system (1.1) are isolated, they can be deformed to the case of generic $z_1,\dots,z_n$. Now the theorem follows from the last lemma and Theorem 1.2 for generic $z_1,\dots,z_n$. □

Remark. There is another proof to claim b) of Theorem 1.2. We give it below.

(1.15) Lemma. *Let* κ *be generic. Let* $z_1,\dots,z_n$ *be well separated. For any unadmissible offdiagonal solution* $t_1,\dots,t_\ell$ *to system* (1.1) *the set* $\{\,t_1,\dots,t_\ell\,\}$ *contains a set* $\{\,z_m-\Lambda_m,\,z_m-\Lambda_m+1,,\dots,$ $z_m+\Lambda_m\,\}$ *for some* m.

Proof. Already established claims a) and c) of Theorem 1.2 imply that it suffices to prove the lemma only for generic $z_1,\dots,z_n$. In the last case any offdiagonal solution to system (1.1) belongs to the orbit of solution $t(\nu,\kappa)$ for a suitable ν. Hence, the lemma follows from the proof to Lemma 1.11. □

(1.16) Theorem. [T] (cf. [CP1] for more detailed proof) *Let σ be a permutation of $1,\dots,n$. Set $V^\sigma = V_{\sigma(1)} \otimes \ldots \otimes V_{\sigma(n)}$ and $z^\sigma = (z_{\sigma(1)},\dots,z_{\sigma(n)})$. Let $w^\sigma(t)$ be constructed in the same manner for V^σ, z^σ as $w(t)$ is constructed for V, z. Then for well separated $z_1,\dots,z_n$ there is a linear isomorphism $V \to V^\sigma$ such that $w(t) \mapsto w^\sigma(t)$.*

By Lemma 1.15 and taking into account Theorem 1.16 we can assume that $\tilde t_1 = z_1 - \Lambda_1$, $\tilde t_2 = z_1 - \Lambda_1 + 1$, $\dots$, $\tilde t_{2\Lambda_1+1} = z_1 + \Lambda_1$ without loss of generality. The structure of the formula in Lemma 1.12 is such that the scalar factor is zero unless $\tilde t_m \in \Gamma_1$, $m = 1,\dots,2\Lambda+1$. Hence, only terms with $\#\Gamma_1 \geqslant 2\Lambda_1 + 1$ survive in the sum. But in these terms $F^{\nu(\Gamma)} = 0$ since $f^{2\Lambda_1+1}v_1 = 0$. Therefore, $w(\tilde t) = 0$. □

2. Bases of Bethe vectors in $\mathfrak{sl}_2$-modules; bases of singular vectors

In this section we always assume that $\kappa = 1$. Set $\operatorname{Sing} V = \{ v \in V \mid \delta(e)\cdot v = 0 \}$ and $\operatorname{Sing} V_{[\ell]} = V_{[\ell]} \cap \operatorname{Sing} V$. Let $\mathfrak{C}_\circ$ be the set of $\mathbf{S}_\ell$-orbits of nontrivial isolated admissible offdiagonal solutions to system (1.1).

(2.1) Lemma. [F], [FT2] *Let t be an offdiagonal solution to system* (1.1). *Then* $w(t) \in \operatorname{Sing} V_{[\ell]}$.

(2.2) Theorem. *Let $\kappa = 1$. Then*

a) *For generic $z_1,\dots,z_n$, all nontrivial isolated admissible offdiagonal solutions to system* (1.1) *are nondegenerate.*

b) *For any $z_1,\dots,z_n$, all trivial admissible offdiagonal solutions to system* (1.1) *are degenerate.*

c) *For generic $z_1,\dots,z_n$, all isolated unadmissible offdiagonal solutions to system* (1.1) *are trivial.*

d) *For generic $z_1,\dots,z_n$, $\#\mathfrak{C}_\circ = \dim \operatorname{Sing} V_{[\ell]}$ and the corresponding Bethe vectors form a base in* $\operatorname{Sing} V_{[\ell]}$.

Proof. A trivial admissible solution to system (1.1) is either nonisolated, and hence degenerate, or degenerate by Theorem 1.13. This proves claim b).

Let $s \in \mathbb{C}$, $s \neq 0$. Make a change of variables $x = sz \in \mathbb{C}^n$, $u = st \in \mathbb{C}^\ell$. In the new variables $u_1, \dots, u_\ell$ system (1.1) reads as follows:

$$\prod_{m=1}^{n} (t_a - z_m + s\Lambda_m) \prod_{\substack{b=1\\ b\neq a}}^{\ell} (t_a - t_b - s) = \kappa \prod_{m=1}^{n} (t_a - z_m - s\Lambda_m) \prod_{\substack{b=1\\ b\neq a}}^{\ell} (t_a - t_b + s)\,. \tag{2.1}$$

$a = 1,\dots,\ell$. As $s \to 0$ system (2.1) turns into

$$\prod_{m=1}^{n} (u_a - x_m) \prod_{\substack{b=1\\ b\neq a}}^{\ell} (u_a - u_b) \Bigl(\sum_{m=1}^{n} \frac{\Lambda_m}{u_a - x_m} - \sum_{\substack{b=1\\ b\neq a}}^{\ell} \frac{1}{u_a - u_b} \Bigr) = 0\,, \tag{2.2}$$

$a = 1,\dots,\ell$. Both systems (2.1) and (2.2) are preserved by the natural action of the symmetric group $\mathbf{S}_\ell$ on variables $u_1,\dots,u_\ell$.

(2.3) Lemma. [RV] *Let $x_1,\dots,x_n$ be generic. Then there are* $\dim \operatorname{Sing} V_{[\ell]}$ *$\mathbf{S}_\ell$-orbits of nondegenerate offdiagonal solutions to system* (2.2).

Let $\mathfrak{C}^\circ$ be the set of $\mathbf{S}_\ell$-orbits of nondegenerate admissible offdiagonal solutions to system (1.1). Since system (1.1) is a deformation of system (2.2), then Lemma 2.3 means that $\#\mathfrak{C}^\circ \geqslant \dim \operatorname{Sing} V_{[\ell]}$ for generic $z_1,\dots,z_n$. On the other hand Lemma 2.1 and Theorem 1.13 show $\#\mathfrak{C}^\circ \leqslant \dim \operatorname{Sing} V_{[\ell]}$. Then $\#\mathfrak{C}^\circ = \dim \operatorname{Sing} V_{[\ell]}$. Moreover, the corresponding Bethe vectors form a base in $\operatorname{Sing} V_{[\ell]}$.

Let $\tilde t$ be a solution to system (1.1) such that $\mathbf{S}_\ell(\tilde t) \in \mathfrak{C}_\circ \setminus \mathfrak{C}^\circ$. Since $\{ w(t) \mid \mathbf{S}_\ell(t) \in \mathfrak{C}^\circ \}$ is a base in $\operatorname{Sing} V_{[\ell]}$ Theorem 1.13 implies that $w(\tilde t) = 0$, and hence, $\mathfrak{C}_\circ \subset \mathfrak{C}^\circ$. The opposite inclusion $\mathfrak{C}^\circ \subset \mathfrak{C}_\circ$ clearly follows from claim b), and we obtain that $\mathfrak{C}_\circ = \mathfrak{C}^\circ$.

If $\tilde t$ is an isolated unadmissible offdiagonal solution to system (1.1) then Theorem 1.13 again implies that $w(\tilde t) = 0$ since $\{ w(t) \mid \mathbf{S}_\ell(t) \in \mathfrak{C}^\circ \}$ is a base in $\operatorname{Sing} V_{[\ell]}$. The theorem is proved. □

(2.4) Theorem. *Let* $\kappa = 1$. *Let* $z_1, \ldots, z_n$ *be generic.*

a) *Let* $2\Lambda_m \notin \mathbb{Z}_{\geqslant 0}$ *for some* m. *Then all degenerate admissible offdiagonal solutions to system* (1.1) *are nonisolated.*

b) *Let* $2\Lambda_m \in \mathbb{Z}_{\geqslant 0}$, $m = 1, \ldots, n$. *Then all degenerate or unadmissible offdiagonal solutions to system* (1.1) *are trivial.*

Proof. Assume, for example that $2\Lambda_1 \notin \mathbb{Z}_{\geqslant 0}$. All other cases can be considered similarly. Let $t \notin \mathfrak{Z}$. By Lemma 1.12 we have

$$w(t) = f^{\ell} v_1 \otimes v_2 \otimes \ldots \otimes v_n \prod_{a=1}^{\ell} (t_a - z_1 - \Lambda_1) + \sum_{k=0}^{\ell-1} f^k v_1 \otimes w_k(t) \neq 0 \,.$$

Here $w_k(t)$ are suitable $V_2 \otimes \ldots \otimes V_n$-valued polynomials. Since by Theorem 2.2 for generic $z_1, \ldots, z_n$ any isolated degenerate admissible offdiagonal solution to system (1.1) is trivial, claim a) is proved.

By Theorem 2.2 any degenerate or unadmissible offdiagonal solution to system (1.1) is either trivial or nonisolated. By Lemma 1.4 existence of nonisolated offdiagonal solutions implies that $\ell \geqslant \Lambda + 1$. But $\operatorname{Sing} V_{[\ell]} = 0$ for $\ell > \Lambda$ in the case in question. Lemma 2.1 completes the proof. □

3. Difference equations with regular singular points. Additive case

Consider a second order differential equation with polynomial coefficients

$$L(u)\xi''(u) + M(u)\xi'(u) + N(u)\xi(u) = 0 \,, \tag{3.1}$$

$\deg L = n$, $\deg M = n - 1$. Assume that this equation has n pairwise distinct regular singular points $z_1, \ldots, z_n$ with exponents $0, \lambda_m + 1$, $\lambda_m \in \mathbb{Z}_{>0}$, at a point z_m, $m = 1, \ldots, n$. This means that $L(u) = \prod_{m=1}^{n} (u - z_m)$ and $M(u)/L(u) = -\sum_{m=1}^{n} \lambda_m/(u - z_m)$.

Global problem. To determine a polynomial $N(u)$ such that equation (3.1) has a nonzero polynomial solution.

Local problem. To determine a polynomial $N(u)$, $\deg N \leqslant n - 2$, such that all solutions to equation (3.1) are entire functions.

These problems arise in separation of variables in the Gaudin model [S1], see also [Sz, Sect. 6.8], [RV]. The next theorem, first observed by Sklyanin [S1], easily follows from analytic theory of differential equations.

(3.1) Theorem. *Let* $N(u)$ *be a solution to the local problem. Then* $N(u)$ *is a solution to the global problem.*

Problem. For fixed $\lambda_1, \ldots, \lambda_n$ and $z_1, \ldots, z_n$ to determine the number of solutions to the formulated problems, see [S1], [Sz, Sect. 6.8], [RV].

In this section we consider similar problems for difference equations arising in separation of variables in quantum lattice integrable models [S2], [S3]. Similar difference equations were introduced by Baxter in his famous studies of two-dimensional exactly solvable models in statistical mechanics [B].

We continue to use notations of previous sections. All over this section we assume that $2\Lambda_m \in \mathbb{Z}_{\geqslant 0}$, $m = 1, \ldots, n$. We also assume that $z_1, \ldots, z_n$ are well separated, which means that all points

$$z_m - \Lambda_m + s \,, \qquad s = 0, \ldots, 2\Lambda_m \,, \quad m = 1, \ldots, n \,,$$

are pairwise distinct.

Consider the second order difference equation

$$\tau(u)Q(u) = Q(u-1) \prod_{m=1}^{n} (u - z_m + \Lambda_m) + \kappa\, Q(u+1) \prod_{m=1}^{n} (u - z_m - \Lambda_m) \,. \tag{3.2}$$

with respect to $Q(u)$. Here κ is a fixed complex number.

Let $\mathcal{S}_m = \{\, z_m - \Lambda_m + s \mid s = 0, \ldots, 2\Lambda_m \,\}$, $m = 1, \ldots, n$. Set $\mathcal{S} = \bigcup_{m=1}^{n} \mathcal{S}_m$. Let $\mathcal{F}_m = \{\, f : \mathcal{S}_m \to \mathbb{C} \,\}$ and let $\mathcal{F} = \{\, f : \mathcal{S} \to \mathbb{C} \,\}$. Let $\pi_m : \mathcal{F} \to \mathcal{F}_m$ be the canonical projection: $\pi_m \varphi = \varphi\big|_{\mathcal{S}_m}$.

We consider the next problems related to difference equation (3.2).

Global problem, $\kappa \neq 1$. To determine a polynomial $\tau(u)$ such that there exists a nontrivial polynomial solution to equation (3.2).

Global problem, $\kappa = 1$. To determine a polynomial $\tau(u)$ such that there exists a nontrivial polynomial solution to equation (3.2) of degree at most $1/2 + \sum_{m=1}^{n} \Lambda_m$.

Local problem. To determine a polynomial $\tau(u) = (1+\kappa)\, u^n + \ldots$ of degree n such that there is $Q \in \mathcal{F}$, $\pi_m Q \neq 0$, $m = 1, \ldots, n$, satisfying equation (3.2) for all $u \in \mathcal{S}$.

Remark. The specification of the global problem for $\kappa = 1$ is motivated by Lemma 3.3.

Remark. Note that a given $Q \in \mathcal{F}$ can satisfy equation (3.2) for at most one polynomial $\tau(u) = (1+\kappa)\, u^n + \ldots$ of degree n, since sets $\mathcal{S}_m$, $m = 1, \ldots, n$, are pairwise disjunctive.

In this section we show (Theorems 3.4, 3.5) that solution to the local problem is also a solution to the global problem and count the number of local solutions.

Remark. Write equation (3.2) in the form

$$\tau(u) Q(u) = \Delta^{+}(u) Q(u-1) + \Delta^{-}(u) Q(u+1) \tag{3.3}$$

and restrict it to $\mathcal{S}_m$. Then we have a finite-dimensional homogeneous system of linear equations

$$\begin{aligned}
\tau(z_m - \Lambda_m) Q(z_m - \Lambda_m) &= \Delta^{-}(z_m - \Lambda_m) Q(z_m - \Lambda_m + 1)\,, \\
\tau(z_m - \Lambda_m + s) Q(z_m - \Lambda_m + s) &= \Delta^{+}(z_m - \Lambda_m + s) Q(z_m - \Lambda_m + s - 1) + {} \\
&\quad + \Delta^{-}(z_m - \Lambda_m + s) Q(z_m - \Lambda_m + s + 1)\,, \\
\tau(z_m + \Lambda_m) Q(z_m + \Lambda_m) &= \Delta^{+}(z_m + \Lambda_m) Q(z_m + \Lambda_m - 1)\,,
\end{aligned} \tag{3.4}$$

$s = 1, \ldots, 2\Lambda_m - 1$. This system has a nontrivial solution if its determinant is zero, that is if $\tau\big|_{\mathcal{S}_m}$ satisfies one additional equation.

Equation (3.2) motivates the following definition for a general difference equations of form (3.3): Say that equation (3.3) has a regular singular point at z with exponents $0, \lambda$ for $\lambda \in \mathbb{Z}_{\geqslant 0}$, if $\Delta^{+}(z - \lambda/2) = \Delta^{-}(z + \lambda/2) = 0$. It would be interesting if this notion could lead to a difference analog of the theory of differential equations with regular singular points.

Now let us return to the local and global problems.

Let $Q \in \mathcal{F}$. Say that Q is a *pseudoconstant* if all projections $\pi_m Q$ are constant functions.

(3.2) Lemma. *Let $\tau(u)$ be a solution to the local problem. Then a solution $Q \in \mathcal{F}$ to equation* (3.2) *is unique modulo a pseudoconstant factor.*

Proof. It is clear that equation (3.2) splits into n independent equations for projections $\pi_1 Q, \ldots, \pi_n Q$. The equation for projection $\pi_m Q$ is system (3.4). Since, any two of solutions to system (3.4) are proportional, the lemma is proved. □

(3.3) Lemma. *Let $\tau(u)$ be a solution to the global problem. Then*

a) $\deg \tau = n$ *and* $\tau(u) = (1+\kappa)\, u^n + \ldots$.

b) *For a given $\tau(u)$, any two of the required polynomial solutions to equation* (3.2) *are proportional.*

Proof. Let $\tau(u) = \sum_{k=0}^{s} a_k u^{s-k}$, $a_0 \neq 0$. Let $Q(u) = \sum_{k=0}^{\ell} b_k u^{\ell-k}$, $b_0 \neq 0$, be a required polynomial solution to equation (3.2). Set $\Lambda = \sum_{m=1}^{n} \Lambda_m$. If $\kappa \neq 1$, then equation (3.2) implies claim a) as well as

$$\ell = \Lambda - \sum_{m=1}^{n} z_m (1+\kappa)/(1-\kappa) - a_1/(1-\kappa)\,, \tag{3.5}$$

$$k\, b_k = b_0 F_k(a_1, \ldots, a_{k+1}, b_1, \ldots, b_{k-1})\,, \qquad k = 1, \ldots, \ell\,.$$

If $\kappa = 1$, then equation (3.2) implies claim a) as well as

$$\ell(\ell - 2\Lambda - 1) = a_2 + \sum_{k=2}^{n} \sum_{m=1}^{k-1} (z_k z_m + \Lambda_k \Lambda_m), \tag{3.6}$$

$$k(2\ell - 2\Lambda - k - 1)\, b_k = b_0 F_k(a_2, \ldots, a_{k+2}, b_1, \ldots, b_{k-1}), \qquad k = 1, \ldots, \ell.$$

Both relations (3.5) and (3.6) clearly fix $Q(u)$ modulo a constant factor, since for $\kappa = 1$ we assume that $\ell \leqslant \Lambda + 1/2$. □

Later on, if $\tau(u)$ is a solution to the global (local) problem and $Q(u)$ is the corresponding solution to equation (3.2), then we also say that $\tau(u)$, $Q(u)$ is a global (local) solution.

Let $\tau(u)$, $Q(u)$ be a solution to the global problem. If $\tau(u)$ is also a solution to the local problem (that is if $Q\big|_{\mathcal{S}_m} \neq 0$, $m = 1, \ldots, n$), then say that $\tau(u)$ is an *admissible* global solution.

Let $\mathfrak{g} = \mathfrak{sl}_2$. Let $V_1, \ldots, V_n$ be irreducible highest weight $\mathfrak{g}$-modules with highest weights $\Lambda_1, \ldots, \Lambda_n$, respectively. Set $V = V_1 \otimes \ldots \otimes V_n$. Let $\operatorname{Sing} V$ be the subspace of singular vectors in V.

(3.4) Theorem. *Let κ be generic. Let $z_1, \ldots, z_n$ be well separated. Then*
a) *All solutions to the local problem are also solutions to the global problem.*
b) *The number of solutions to the local problem is equal to* $\dim V$.
c) *If $\tau(u)$, $Q(u)$ is an admissible solution to the global problem, then* $\deg Q \leqslant 2 \sum_{m=1}^{n} \Lambda_m$.

(3.5) Theorem. *Let $\kappa = 1$. Let $z_1, \ldots, z_n$ be generic. Then*
a) *All solutions to the local problem are also solutions to the global problem.*
b) *The number of solutions to the local problem is equal to* $\dim V$.
c) *If $\tau(u)$, $Q(u)$ is an admissible solution to the global problem, then* $\deg Q \leqslant \sum_{m=1}^{n} \Lambda_m$.

We prove these theorems in two steps. First we obtain the required number of admissible global solutions. Next we show that the number of local solutions cannot exceed $\dim V$ or $\dim \operatorname{Sing} V$, respectively.

(3.6) Lemma. *Let $z_1, \ldots, z_n$ be well separated. Let $t_1, \ldots, t_\ell$ be an admissible offdiagonal solution to system* (1.1). *Let $\tau(u) = \tau(u, t)$ be given by formula* (1.2). *Set $Q(u) = \prod_{a=1}^{\ell} (u - t_a)$. Then $\tau(u)$, $Q(u)$ is an admissible solution to the global problem.*

Proof. By (1.2) $\tau(u)$ is a rational function in u with only simple poles at points $t_1, \ldots, t_\ell$. System (1.1) means that $\operatorname{Res}_{u=t_a} \tau(u) = 0$, hence $\tau(u)$ is a polynomial. Equation (3.2) is fulfilled by the definition of $\tau(u)$, $Q(u)$. Since for $\kappa = 1$, $\ell > 1/2 + \sum_{m=1}^{n} \Lambda_m$ there are no admissible offdiagonal solutions to system (1.1), we have $\deg Q \leqslant 1/2 + \sum_{m=1}^{n} \Lambda_m$ and hence $\tau(u)$, $Q(u)$ is a solution to the global problem. This global solution is clearly admissible since $Q(u \pm \Lambda_m) \neq 0$, $m = 1, \ldots, n$, by the definition of an admissible solution $t_1, \ldots, t_\ell$ to system (1.1). □

(3.7) Lemma. *Let M be the total number of $\mathbf{S}_\ell$-orbits of admissible solutions to system* (1.1) *for $\ell = 0, \ldots, 2 \sum_{m=1}^{n} \Lambda_m$ altogether.*
a) *Let κ be generic. Let $z_1, \ldots, z_n$ be well separated. Then $M = \dim V$. Moreover, there are no admissible offdiagonal solutions to system* (1.1) *for $\ell > 2 \sum_{m=1}^{n} \Lambda_m$.*
b) *Let $\kappa = 1$. Let $z_1, \ldots, z_n$ be generic. Then $M = \dim \operatorname{Sing} V$. Moreover, there are no admissible offdiagonal solutions to system* (1.1) *for $\ell > \sum_{m=1}^{n} \Lambda_m$.*

Proof. Claim a) follows from Theorem 1.2. Claim b) follows from Theorems 2.2 and 2.4. □

Lemmas 3.6 and 3.7 give the required number of admissible solutions to the global problem. To get an estimate from above for the number of local solutions we consider a spectral problem which can be

solved by separation of variables. Equation (3.2) is the equation for separated variables in this problem. All constructions below are motivated by the functional Bethe ansatz [S2].

Let $\mathcal{F}_\otimes = \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$. We consider $\mathcal{F}_\otimes$ as a space of functions in n variables $x_1 \in \mathcal{S}_1, \ldots, x_n \in \mathcal{S}_n$. Let $y_k^\pm \in \mathrm{End}(\mathcal{F}_\otimes)$, $k = 1, \ldots, n$, be defined as follows:

$$y_k^\pm f(x_1, \ldots, x_n) = f(x_1, \ldots, x_k \mp 1, \ldots, x_n) \prod_{m=1}^{n} (x_k - z_m \pm \Lambda_m).$$

Set

$$\mathcal{T}(u) = (1+\kappa) \prod_{m=1}^{n} (u - x_m) + \sum_{m=1}^{n} \prod_{\substack{k=1\\k\neq m}}^{n} \frac{u - x_k}{x_m - x_k} \cdot (y_m^+ + \kappa y_m^-).$$

(3.8) Lemma. [S2, Sect. 2.4] *Coefficients of the polynomial $\mathcal{T}(u)$ generate a commutative subalgera in* $\mathrm{End}(\mathcal{F}_\otimes)$.

The proof is straightforward.

(3.9) Lemma. [S2, Sect. 2.6] *Let $\tau(u)$, $Q(u)$ be a solution to the local problem. Set $Q_\otimes = \pi_1 Q \otimes \ldots \otimes \pi_n Q \neq 0$. Then $\mathcal{T}(u) Q_\otimes = \tau(u) Q_\otimes$. Moreover, any eigenvector of $\mathcal{T}(u)$ has the form $Q_\otimes$ for a suitable solution $\tau(u)$, $Q(u)$ to the local problem.*

Proof. The key observation for the proof of the lemma is that for any $f \in \mathcal{F}_\otimes$

$$\begin{aligned}\mathcal{T}(x_k) f(x_1, \ldots, x_n) = f(x_1, \ldots, x_k - 1, \ldots, x_n) \prod_{m=1}^{n} (x_k - z_m + \Lambda_m) +\\ + \kappa f(x_1, \ldots, x_k + 1, \ldots, x_n) \prod_{m=1}^{n} (x_k - z_m - \Lambda_m).\end{aligned} \tag{3.7}$$

The first claim of the lemma means that for any $(x_1, \ldots, x_n) \in \mathcal{S}_1 \times \ldots \times \mathcal{S}_n$

$$\big(\mathcal{T}(u) Q_\otimes - \tau(u) Q_\otimes\big)(x_1, \ldots, x_n) = 0. \tag{3.8}$$

Fix $x_1 \in \mathcal{S}_1, \ldots, x_n \in \mathcal{S}_n$. Now the left hand side above is a polynomial in u of degree $n-1$ and it suffices to show that it is zero for n pairwise distinct values of u. Consider points $x_1, \ldots, x_n$ which are pairwise distinct, since $x_m \in \mathcal{S}_m$. Substitution $u = x_m$ to (3.8) reduces it by means of (3.7) to equation (3.2) for projection $\pi_m Q$ at $u = x_m$, which is valid.

Let $f \in \mathcal{F}_\otimes$ be an eigenvector of $\mathcal{T}(u)$ with an eigenvalue $\tau(u)$:

$$\mathcal{T}(u) f(x_1, \ldots, x_n) = \tau(u) f(x_1, \ldots, x_n). \tag{3.9}$$

Then $\tau(u)$ is a polynomial in u of degree n and $\tau(u) = (1+\kappa) u^n + \ldots$. Substituting $u = x_1$ to equation (3.9) we get that $\tau(x_1)$, $f(x_1, x_2, \ldots, x_n)$ satisfy equation (3.2) with respect to the variable $x_1 \in \mathcal{S}_1$. Similar to the proof of Lemma (3.2) we obtain that $f(x_1, \ldots, x_n) = f_1(x_1) \otimes f_{[1]}(x_2, \ldots, x_n)$ for $f_1 \in \mathcal{F}_1$ which obeys equation (3.2) and suitable $f_{[1]} \in \mathcal{F}_2 \otimes \ldots \otimes \mathcal{F}_n$. Similarly, $\tau(x_2)$, $f_{[1]}(x_2, x_3, \ldots, x_n)$ satisfy equation (3.2) with respect to a variable $x_2 \in \mathcal{S}_2$ and $f_{[1]}(x_2, \ldots, x_n) = f_2(x_2) \otimes f_{[2]}(x_3, \ldots, x_n)$ for suitable f_2, $f_{[2]}$, etc. Finally, $f(x_1, \ldots, x_n) = f_1(x_1) \otimes \ldots \otimes f_n(x_n)$ and $\tau(u)$, $f_m(u)$ satisfy equation (3.2) for $u \in \mathcal{S}_m$, $m = 1, \ldots, n$. Since $\mathcal{S}_k \cap \mathcal{S}_m = \varnothing$ unless $k = m$, there is a solution $\tau(u)$, $Q(u)$ to the local problem, such that $f_1 = \pi_1 Q, \ldots, f_n = \pi_n Q$ which means $f = Q_\otimes$. □

Proof of Theorem 3.4. By Lemmas 3.6, 3.7 and 3.3 we point out $\dim V$ admissible solutions $\tau(u)$, $Q(u)$ to the global problem. For all of them $\deg Q(u) \leqslant 2 \sum_{m=1}^{n} \Lambda_m$. Since any admissible global solution is also a local solution, Lemma 3.9 shows that we exhaust all local as well as admissible global solutions. □

From now until the end of the section, let $\kappa = 1$. To get the required estimate from above for the number of solutions to the local problem in this case we equip the space $\mathcal{F}_\otimes$ with a structure of a

$\mathfrak{g}$-module isomorphic to the $\mathfrak{g}$-module V. Set

$$H = \sum_{m=1}^{n} \Bigl(x_m - z_m - \prod_{\substack{k=1 \\ k\neq m}}^{n} (x_m - x_k)^{-1} \cdot y_m^- \Bigr) ,$$

$$F = \sum_{m=1}^{n} \prod_{\substack{k=1 \\ k\neq m}}^{n} (x_m - x_k)^{-1} \cdot y_m^- ,$$

$$E = \sum_{m=1}^{n} \Bigl(2(x_m - z_m) - \prod_{\substack{k=1 \\ k\neq m}}^{n} (x_m - x_k)^{-1} \cdot (y_m^+ + y_m^-) \Bigr) ,$$

$$\Delta(u) = \prod_{m=1}^{n} (u - z_m + \Lambda_m)(u - z_m - \Lambda_m - 1) .$$

(3.10) Lemma. *The map* $e \to E$, $f \to F$, $h \to H$ *makes the space* $\mathcal{F}_\otimes$ *into a* $\mathfrak{g}$*-module isomorphic to the* $\mathfrak{g}$*-module* V.

Proof. Verifying commutation relations between E, F, H is cumbersome, but straightforward. The following identity is useful:

$$\sum_{m=1}^{n} \Bigl(\Delta(x_m + 1) \prod_{\substack{k=1 \\ k\neq m}}^{n} \frac{1}{(x_m - x_k)(x_m - x_k + 1)} - {}$$

$$- \Delta(x_m) \prod_{\substack{k=1 \\ k\neq m}}^{n} \frac{1}{(x_m - x_k - 1)(x_m - x_k)} \Bigr) = 2 \sum_{m=1}^{n} (x_m - z_m) .$$

The identity itself is equivalent to

$$\Bigl(\operatorname*{Res}_{u=\infty} + \sum_{m=1}^{n} \bigl(\operatorname*{Res}_{u=x_m} + \operatorname*{Res}_{u=x_m+1} \bigr) \Bigr) \Delta(u) \prod_{m=1}^{n} \frac{1}{(u - x_m)(u - x_m - 1)} = 0 .$$

The character of the obtained $\mathfrak{g}$-module $\mathcal{F}_\otimes$ coincides with the character of the $\mathfrak{g}$-module V since dimensions of the corresponding weight subspaces are clearly the same. This proves the required isomorphism. □

Define one more polynomial taking values in $\operatorname{End}(\mathcal{F}_\otimes)$:

$$\mathcal{C}(u) = \sum_{m=1}^{n} \Bigl(\prod_{\substack{k=1 \\ k\neq m}}^{n} \frac{u - x_k}{x_m - x_k} \cdot \Bigl(\Delta(x_m + 1) \prod_{k=1}^{n} (x_m - x_k + 1)^{-1} + {}$$

$$+ \Delta(x_m) \prod_{k=1}^{n} (x_m - x_k - 1)^{-1} - y_m^+ - y_m^- \Bigr) + {}$$

$$+ \sum_{\substack{l=1 \\ l\neq m}}^{n} (x_m - x_l)^{-1} (x_m - x_l - 1)^{-1} \prod_{\substack{k=1 \\ k\neq l,m}}^{n} \frac{u - x_k}{(x_m - x_k)(x_l - x_k)} \cdot y_m^+ y_l^- \Bigr) .$$

The next lemma is proved by tremendous, but all the same straightforward calculation.

(3.11) Lemma.

a) *Coefficients of the polynomial* $\mathcal{T}(u) + \mathcal{C}(u)$ *generate a commutative subalgera in* $\operatorname{End}(\mathcal{F}_\otimes)$.
b) *Coefficients of the polynomial* $\mathcal{T}(u) + \mathcal{C}(u)$ *commute with the* $\mathfrak{sl}_2$ *action in* $\mathcal{F}_\otimes$.
c) *Coefficients of the polynomial* $\mathcal{C}(u)$ *are raising operators:* $[H, \mathcal{C}(u)] = H$.

Proof of Theorem 3.5. By Lemmas 3.10 and 3.11 $\mathcal{T}(u) + \mathcal{C}(u)$ can have at most $\dim \operatorname{Sing} V$ different eigenvalues and the same is true for $\mathcal{T}(u)$. Hence, by Lemma 3.9 there are at most $\dim \operatorname{Sing} V$ solutions to the local problem.

On the other hand, by Lemmas 3.6, 3.7 and 3.3 we point out $\dim \operatorname{Sing} V$ admissible solutions $\tau(u)$, $Q(u)$ to the global problem. For all of them $\deg Q(u) \leqslant \sum_{m=1}^{n} \Lambda_m$. Since any admissible global solution is also a local solution, we exhaust all local as well as admissible global solutions. □

4. Bases of Bethe vectors in $U_q(\mathfrak{sl}_2)$-modules

In this section we describe a q-variant of Theorem 1.2. Proofs of these new statements are completely similar to the corresponding proofs of Section 1. Notations, used in this section differ slightly from the notations, used in Section 1.

Let $\mathbb{C}_\circ = \mathbb{C} \setminus \{0\}$. Let $q \in \mathbb{C}_\circ$, $q^2 \neq 1$. Let $\mathfrak{g} = \mathfrak{sl}_2$. Let e, f, q^h, q^{-h} be generators of $U_q(\mathfrak{g})$:

$$q^h e q^{-h} = qe\,, \qquad q^h f q^{-h} = q^{-1} f\,, \qquad [e,f] = \frac{q^{2h} - q^{-2h}}{q - q^{-1}}\,.$$

Let $M = \mathrm{End}\,(\mathbb{C}^2)$. Introduce $T(u) \in M[u] \otimes U_q(\mathfrak{g})$ as follows:

$$T(u) = \begin{pmatrix} uq^h - q^{-h} & uf\,(q - q^{-1}) \\ e\,(q - q^{-1}) & uq^{-h} - q^h \end{pmatrix}.$$

Let ι_m be the embedding $U_q(\mathfrak{g}) \to U_q(\mathfrak{g})^{\otimes n}$ as the m-th tensor factor.

Let $z = (z_1, \ldots, z_n) \in \mathbb{C}_\circ^n$. Let $T_m(u) = \mathrm{id} \otimes \iota_m\bigl(T(u)\bigr) \in M[u] \otimes U_q(\mathfrak{g})^{\otimes n}$. Set

$$\mathrm{T}(u) = z_1 T_1(u/z_1) \ldots z_n T_n(u/z_n) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

where $A(u)$, $B(u)$, $C(u)$ and $D(u)$ are suitable elements in $\mathbb{C}\,[u] \otimes U(\mathfrak{g})^{\otimes n}$.

Let $\kappa \in \mathbb{C}$. Set $\mathcal{T}(u) = A(u) + \kappa D(u)$. Coefficients of the polynomial $\mathcal{T}(u)$ generate a remarkable commutative subalgebra in $U(\mathfrak{g})^{\otimes n}$.

Remark. From the physical point of view the q-deformed case considered in this section corresponds to partially anisotropic quantum models, while the previous case corresponds to isotropic quantum models.

Later on in this section we assume that q is not a root of unity. In the next section we partially extend results of this section to the case when q is a root of unity.

Let $V_1, \ldots, V_n$ be irreducible highest weight $U_q(\mathfrak{g})$-modules with highest weights $\Lambda_1, \ldots, \Lambda_n$ and generating vectors $v_1, \ldots, v_n$, respectively. Set $V = V_1 \otimes \ldots \otimes V_n$. Set $\Lambda = \sum_{m=1}^{n} \Lambda_m$. Let $\ell \in \mathbb{Z}_{\geqslant 0}$. Let $V_{[\ell]} \subset V$ be a weight subspace $V_{[\ell]} = \{\, v \in V \mid q^h \otimes \ldots \otimes q^h \cdot v = q^{\Lambda - \ell} v \,\}$.

Remark. All the time only $q^{\Lambda_1}, \ldots, q^{\Lambda_n}$ are used. $\Lambda_1, \ldots, \Lambda_n$ themselves never appear in formulae.

Let $t = (t_1, \ldots, t_\ell) \in \mathbb{C}_\circ^\ell$. Set

$$w(t) = B(t_1) \ldots B(t_\ell) \cdot v_1 \otimes \ldots \otimes v_n\,. \tag{4.1}$$

$w(t)$ is a $V_{[\ell]}$-valued symmetric polynomial in variables $t_1, \ldots, t_\ell$. Another formula for $w(t)$ see below in Lemma 4.10.

For given $z_1, \ldots, z_n$ consider a system of algebraic equations on variables $t_1, \ldots, t_\ell$:

$$\prod_{m=1}^{n} (q^{2\Lambda_m} t_a - z_m) \prod_{\substack{b=1 \\ b \neq a}}^{\ell} (t_a - q^2 t_b) = \kappa \prod_{m=1}^{n} (t_a - q^{2\Lambda_m} z_m) \prod_{\substack{b=1 \\ b \neq a}}^{\ell} (q^2 t_a - t_b)\,. \tag{4.2}$$

$a = 1, \ldots, \ell$. We consider this system only in $\mathbb{C}_\circ^\ell$. A solution t to system (4.2) is called an *offdiagonal* solution if are pairwise distinct, and a *diagonal* solution, otherwise.

Set

$$\tau(u,t) = q^{-\Lambda - \ell} \Bigl(\prod_{m=1}^{n} (q^{2\Lambda_m} u - z_m) \prod_{a=1}^{\ell} \frac{u - q^2 t_a}{u - t_a} + \kappa \prod_{m=1}^{n} (u - q^{2\Lambda_m} z_m) \prod_{a=1}^{\ell} \frac{q^2 u - t_a}{u - t_a} \Bigr). \tag{4.3}$$

(4.1) Theorem. [BIK], [F], [FT] *Let* $t_1, \dots, t_\ell$ *be an offdiagonal solution to system* (4.2). *Then* $\mathcal{T}(u) \cdot w(t) = \tau(u,t)\, w(t)$.

Define a set $\mathfrak{Z}$ by the equation

$$\prod_{a=1}^{\ell} \Bigl(\prod_{m=1}^{n} (q^{2\Lambda_m} t_a - z_m)(t_a - q^{2\Lambda_m} z_m) \prod_{\substack{b=1 \\ b \neq a}}^{\ell} (t_a - q^2 t_b) \Bigr) = 0\,.$$

A solution t to system (4.2) is called an *admissible* solution if $t \notin \mathfrak{Z}$ and an *unadmissible* otherwise. For an admissible offdiagonal solution t the vector $w(t)$ is called the *Bethe vector*. A solution t to system (4.2) is called a *trivial* solution if $w(t) = 0$ and *nontrivial* otherwise.

System (4.2) is preserved by the natural action of the symmetric group $\mathbf{S}_\ell$ on variables $t_1, \dots, t_\ell$. Therefore, $\mathbf{S}_\ell$ acts on solutions to this system. Let $\mathfrak{C}$ be the set of $\mathbf{S}_\ell$-orbits of admissible offdiagonal solutions.

Say that $z_1, \dots, z_n$ are *well separated* if all points $z_m q^{2(s-\Lambda_m)}$, $s \in \mathbb{Z}_{\geqslant 0}$, $s \leqslant \dim V_m - 2$, and $z_m q^{2\Lambda_m}$, $m = 1, \dots, n$, are pairwise distinct.

Remark. $z_1, \dots, z_n$ are well separated if and only if module V enjoys next properties:
- i) V is irreducible with respect to a subalgebra generated by coefficients of polynomials $A(u)$, $B(u)$, $C(u)$, $D(u)$.
- ii) A commutative subalgebra generated by coefficients of $A(u)$ acts in V in a semisimple way.

This follows from results of [T], [NT]. The part "*only if*" also follows from results of the present paper. Note in addition that the action in V of subalgebra generated by coefficients of $A(u)$ has simple spectrum.

(4.2) Theorem. *Let* κ *be generic. Let* $z_1, \dots, z_n$ *be well separated. Then*
- a) *All admissible offdiagonal solutions to system* (4.2) *are nondegenerate.*
- b) *All unadmissible offdiagonal solutions to system* (4.2) *are trivial.*
- c) $\#\mathfrak{C} = \dim V_{[\ell]}$ *and the corresponding Bethe vectors form a base in* $V_{[\ell]}$.

The proof is completely similar to the proof of Theorem 1.2. We give below only the main points of the proof.

Remark. This Theorem was proved in [TV] for q not a root of unity and generic $z_1, \dots, z_n$, $\Lambda_1, \dots, \Lambda_n$. The proof of Theorem 1.2 is similar to the corresponding proof in [TV].

Remark. A theorem similar to Theorem 4.2 for the case $\Lambda_m = 1/2$, $m = 1, \dots, n$, was announced in a recent preprint [LS].

(4.3) Lemma. *Let* $q^{4\Lambda_m} \notin \{\, q^{2s} \mid s \in \mathbb{Z}_{\geqslant 0} \,\}$, $m = 1, \dots, n$. *For* $\kappa \neq 0$ *and well separated* $z_1, \dots, z_n$ *all solutions to system* (1.1) *are admissible.*

(4.4) Lemma. *System* (4.2) *has no nonisolated offdiagonal solutions unless* $\kappa = q^{2(s-\ell+\Lambda)} = q^{2(\ell-\tilde{s}-\Lambda)}$ *for some* $s, \tilde{s} \in \{\, 1, \dots, \ell \,\}$.

(4.5) Corollary. *System* (4.2) *has no nonisolated offdiagonal solutions unless* $q^{4\Lambda} \in \{\, 1, q^2, \dots, q^{4\ell-4} \,\}$.

(4.6) Lemma. *Consider solutions to system* (4.2) *as (multivalued) functions of* κ.
- a) *For any offdiagonal solution to system* (4.2) *every its branch remains finite for any* $\kappa \neq q^{2(s-\ell+\Lambda)}$, $s = 1, \dots, \ell$.
- b) *For any offdiagonal solution to system* (4.2) *every its branch remains in* $\mathbb{C}^\ell_\circ$ *for any* $\kappa \neq q^{2(\ell-s-\Lambda)}$, $s = 1, \dots, \ell$.

(4.7) Lemma.
- a) *All solutions to system* (4.2) *at* $\kappa = 0$ *are isolated.*
- b) *For generic* $z_1, \dots, z_n$ *all offdiagonal solutions to system* (4.2) *at* $\kappa = 0$ *are nondegenerate.*

(4.8) Lemma. *Let* $z_1, \dots, z_n$ *be generic. Let* $t(\kappa)$ *be a solution to system* (4.2), *which is a deformation of a diagonal solution* $t(0)$ *to system* (4.2) *at* $\kappa = 0$. *Then* $t(\kappa)$ *is a diagonal solution.*

Set $\mathcal{Z}_\ell = \{\, \nu \in \mathbb{Z}^n_{\geqslant 0} \mid \sum_{m=1}^{n} \nu_m = \ell \,\}$. Set $\mathcal{Z}^\circ_\ell = \{\, \nu \in \mathbb{Z}^n_{\geqslant 0} \mid \sum_{m=1}^{n} \nu_m = \ell$ and $\nu_m < \dim V_m$, $m = 1,$

$\ldots, n\}$. For $\nu \in \mathcal{Z}_\ell$ let $t^\star(\nu)$ be the following offdiagonal solution to system (4.2) at $\kappa = 0$

$$t^\star_a(\nu) = q^{2(a-\ell_{i-1}-1-\Lambda_i)} z_i \qquad \text{for } \ell_{i-1} < a \leqslant \ell_i \tag{4.4}$$

where $\ell_i = \sum_{m=1}^{i} \nu_m$, $\ell_0 = 0$, $\ell_n = \ell$. Let $t(\nu, \kappa)$ be a solution to system (4.2) which is a deformation of $t^\star(\nu)$.

(4.9) Lemma. *Let κ be generic.*
a) *Let $\nu \in \mathcal{Z}^\circ_\ell$. Let $z_1, \ldots, z_n$ be well separated. Then $t(\nu, \kappa)$ is an admissible solution.*
b) *Let $\nu \in \mathcal{Z}_\ell \setminus \mathcal{Z}^\circ_\ell$. Let $z_1, \ldots, z_n$ be generic. Then $t(\nu, \kappa)$ is an unadmissible solution.*

Introduce the canonical monomial base in V: $\{ F^\nu = f^{\nu_1} v_1 \otimes \ldots \otimes f^{\nu_n} v_n \}$. It is clear that $\{ F^\nu \mid \nu \in \mathcal{Z}^\circ_\ell \}$ is a base in $V_{[\ell]}$.

(4.10) Lemma. [Ko] *The following decomposition holds*

$$w(t) = \sum_{part} F^{\nu(\Gamma)} \prod_{l=2}^{n} \prod_{m=1}^{l-1} \Bigl(\prod_{\substack{a\in\Gamma_l \\ b\in\Gamma_m}} \frac{q^{-1}t_a - qt_b}{t_a - t_b} \prod_{a\in\Gamma_l} (q^{\Lambda_m} t_a - q^{-\Lambda_m} z_m) \prod_{a\in\Gamma_m} (q^{-\Lambda_m} t_a - q^{\Lambda_m} z_m) \Bigr).$$

Here the sum is taken over all partitions of the set $\{1, \ldots, \ell\}$ into disjoint subsets $\Gamma_1, \ldots, \Gamma_n$ and $\nu(\Gamma) = (\#\Gamma_1, \ldots, \#\Gamma_n)$.

Let v^*_m be a linear function on V_m such that $\langle v^*_m, v_m\rangle = 1$ and $\langle v^*_m, v\rangle = 0$ for any weight vector $v \in V_m$, $v \neq v_m$.

(4.11) Theorem. [Ko], [TV] *Let $\Lambda_1, \ldots, \Lambda_n$, $z_1, \ldots, z_n$ and κ be generic.*
a) *For any offdiagonal solution $t = (t_1, \ldots, t_n)$ to system* (4.2)

$$\begin{aligned} &\langle v^*_1 \otimes \ldots \otimes v^*_n, C(t_1) \ldots C(t_n) w(t)\rangle = \\ &\quad = (-1)^\ell q^{-2\Lambda-\ell(\ell+1)} (q - q^{-1})^\ell \prod_{m=1}^{n} \prod_{a=1}^{\ell} (t_a - q^{2\Lambda_m} z_m) \prod_{a=2}^{\ell} \prod_{b=1}^{a-1} \frac{q^2 t_a - t_b}{t_a - t_b} \times \\ &\quad \times \det \Bigl[t_a \frac{\partial}{\partial t_a} \Bigl(\prod_{m=1}^{n} (q^{2\Lambda_m} t_b - z_m) \prod_{\substack{c=1 \\ c\neq b}}^{\ell} (t_b - q^2 t_c) - \kappa \prod_{m=1}^{n} (t_b - q^{2\Lambda_m} z_m) \prod_{\substack{c=1 \\ c\neq b}}^{\ell} (q^2 t_b - t_c) \Bigr) \Bigr]_{a,b=1,\ldots,\ell}. \end{aligned}$$

b) *For any offdiagonal solutions t and $\tilde t = (\tilde t_1, \ldots, \tilde t_n)$ to system* (1.1) *which lie on different $\mathbf{S}_\ell$-orbits*

$$\langle v^*_1 \otimes \ldots \otimes v^*_n, C(\tilde t_1) \ldots C(\tilde t_n) w(t)\rangle = 0.$$

c) *Let $t, \tilde t$ be isolated solution to system* (1.1). *Then both claims* a) *and* b) *remain valid for any Λ_1, $\ldots, \Lambda_n$, $z_1, \ldots, z_n$ and κ.*

Remark. In this paper we use a normalization of $w(t)$ which differs from the normalization in [TV].

(4.12) Lemma. [T] (cf. [CP2] for more detailed proof) *Let σ be a permutation of $1, \ldots, n$. Set $V^\sigma = V_{\sigma(1)} \otimes \ldots \otimes V_{\sigma(n)}$ and $z^\sigma = (z_{\sigma(1)}, \ldots, z_{\sigma(n)})$. Let $w^\sigma(t)$ be constructed in the same manner for V^σ, z^σ as $w(t)$ is constructed for V, z. Then there is a linear isomorphism $V \to V^\sigma$ such that $w(t) \mapsto w^\sigma(t)$.*

5. Bases of Bethe vectors in $U_q(\mathfrak{sl}_2)$-modules at roots of unity

Let q be a root of unity. In this case a representation theory of $U_q(\mathfrak{g})$ drastically changes (see e.g. [DCK]). Nevertheless, results of the previous section essentially remain valid in this case. We give precise statements in this section.

Later on we keep notations from the previous section unless changes are given explicitly.

Let q^2 be a primitive N-th root of unity. Let V be an irreducible highest weight $\mathfrak{g}$-module with highest weight Λ. In the case in question there are two types of such modules.

I. *Restricted modules.* These modules correspond to $q^{4\Lambda} \in \{1, q^2, , \ldots, q^{2N-4}\}$. A restricted module V is uniquely fixed by its highest weight Λ; $q^{2\dim V} = q^{4\Lambda+2}$, $\dim V < N$. All these modules admit a deformation to the case of generic q.

II. *Unrestricted modules.* These modules correspond to $q^{4\Lambda} \in \mathbb{C}_\circ \setminus \{1, q^2, , \ldots, q^{2N-4}\}$. In this case $\dim V = N$. For a given highest weight Λ there is a one-parametric family of unrestricted modules. They are separated by values of the central element f^N in these modules. The value of f^N in an unrestricted module can be an arbitrary complex number. The only unrestricted modules, which can be deformed to the case of generic q, are those corresponding to $q^{4\Lambda} = q^{-2}$, $f^N = 0$. We call these modules *quasirestricted*.

Let $V_1, \ldots, V_n$ be irreducible highest weight $U_q(\mathfrak{g})$-modules with highest weights $\Lambda_1, \ldots, \Lambda_n$ and generating vectors $v_1, \ldots, v_n$, respectively. Set $V = V_1 \otimes \ldots \otimes V_n$. Set $\Lambda = \sum_{m=1}^{n} \Lambda_m$. Let $\ell \in \mathbb{Z}_{\geqslant 0}$. Let $V_{[\ell]} \subset V$ be the subspace generated by $\{ f^{\nu_1} v_1 \otimes \ldots \otimes f^{\nu_n} v^n \mid \sum_{m=1}^{n} \nu_m = \ell \}$.

Let $t = (t_1, \ldots, t_\ell) \in \mathbb{C}_\circ^\ell$. Let $w(t)$ be defined by (4.1). It is a $V_{[\ell]}$-valued symmetric polynomial in variables $t_1, \ldots, t_\ell$. Let $\tau(u, t)$ be defined by (4.3). Theorem 4.1 is known to remain valid even if q is a root of unity.

Let $\mathfrak{C}$ be the set of $\mathbf{S}_\ell$-orbits of admissible offdiagonal solutions to system (4.2).

Set $s_m = \dim V_m - 2$ if V_m is a restricted or quasirestricted module and $s_m = N - 1$, otherwise. Say that $z_1, \ldots, z_n$ are *well separated* if all points $z_m q^{2(s-\Lambda_m)}$, $s = 0, \ldots, s_m$, and $z_m q^{2\Lambda_m}$, $m = 1, \ldots, n$, are pairwise distinct.

(5.1) Theorem. *Let κ be generic. Let $z_1, \ldots, z_n$ be well separated. Then*
a) *All admissible offdiagonal solutions to system* (4.2) *are nondegenerate.*
b) $\#\mathfrak{C} = \dim V_{[\ell]}$ *and the corresponding Bethe vectors form a base in* $V_{[\ell]}$.

(5.2) Lemma. *Let $\kappa \neq 0$. Then all admissible solutions to system* (4.2) *are isolated provided the couple κ, q^Λ does not belong to a certain finite set.*

Proof. Assume that there is a nonisolated admissible solution to system (4.2). This means that we have a curve $t(s)$, $s \in \mathbb{R}$, such that $t(s)$ is an admissible solution to system (4.2) for any s. Moreover, we can assume that as $s \to +\infty$, $t(s)$ tends to infinity in the following way: $t_a(s) \to \infty$ if $a \leqslant f$ and $t_a(s)$ has a finite limit if $a > f$. Taking the product of the first f equations of system (4.2) we obtain

$$\prod_{a=1}^{f} \prod_{\substack{b=1 \\ b\neq a}}^{\ell} \bigl(t_a(s) - q^2 t_b(s)\bigr) \prod_{a=1}^{f} \Bigl(\prod_{m=1}^{n} \bigl(q^{2\Lambda_m} t_a(s) - z_m\bigr) \prod_{b=f+1}^{\ell} \bigl(t_a(s) - q^2 t_b(s)\bigr) \Bigr) =$$
$$= \kappa^f \prod_{a=1}^{f} \prod_{\substack{b=1 \\ b\neq a}}^{\ell} \bigl(q^2 t_a(s) - t_b(s)\bigr) \prod_{a=1}^{f} \Bigl(\prod_{m=1}^{n} \bigl(t_a(s) - q^{2\Lambda_m} z_m\bigr) \prod_{b=f+1}^{\ell} \bigl(q^2 t_a(s) - t_b(s)\bigr) \Bigr).$$

The first products in the left and right hand sides above coincide. Moreover, they are not zero, since $t(s)$ is an admissible solution. Cancelling these products and taking the limit $s \to \infty$ we obtain that

$$\kappa^f = q^{2f(f-\ell+\Lambda)}. \tag{5.1}$$

System (4.2) is invariant under the transformation $z_m \to z_m^{-1}$, $t_a \to t_a^{-1}$, $\kappa \to \kappa^{-1}$, $m = 1, \ldots, n$, $a = 1, \ldots, \ell$. Therefore we also have that

$$\kappa^{\tilde f} = q^{2\tilde f(\tilde f-\ell+\Lambda)} \tag{5.2}$$

for a suitable $\tilde f \in \{1, \ldots, \ell\}$. Hence, if the couple κ, q^Λ does not obey equations (5.1) and (5.2) for some $f, \tilde f \in \{1, \ldots, \ell\}$ then all admissible solutions to system (4.2) are isolated. □

Proof of Theorem 5.1. Let κ be generic. Then $t^\star(\nu)$, $\nu \in \mathcal{Z}_\ell^\circ$, are nondegenerate offdiagonal solutions to system (4.2) at $\kappa = 0$. Their deformations $t(\nu, \kappa)$, $\nu \in \mathcal{Z}_\ell^\circ$, are admissible nondegenerate offdiagonal solutions to system (4.2) and the corresponding Bethe vectors form a base in $V_{[\ell]}$. The proof is the same as for generic q.

Any unrestricted module can be considered as a continuous deformation of a quasirestricted module. Since for generic κ all admissible solutions are isolated, it suffices to proof the theorem for the case when all $\mathfrak{g}$-modules $V_1, \ldots, V_n$ are restricted or quasirestricted. In the last case, all modules $V_1, \ldots, V_n$ can be deformed to the case of generic q. Then inequality $\#\mathfrak{C} \leqslant \dim V_{[\ell]}$ follows from Theorem 4.2, which completes the proof of Theorem 5.1. □

6. Difference equations with regular singular points. Multiplicative case

In this section we describe a q-variant of Theorem 3.4. Proofs of these new statements are completely similar to the corresponding proofs of Section 3. So we give only the most important points of the proofs. We keep notations used in the two previous sections which differ slightly from the notations, used in Section 3.

Let $q \in \mathbb{C}_\circ$, $q^2 \neq 1$. Details of consideration depend on whether q is or is not a root of unity. We give necessary specifications for the case of q being a root of unity at the end of the section.

Let q be not a root of unity. In this case we assume that $\Lambda_1, \ldots, \Lambda_n$ are such that $q^{4\Lambda_m} = q^{2d_m}$, for some numbers $d_m \in \mathbb{Z}_{\geqslant 0}$, $m = 1, \ldots, n$. Note that the integers $d_1, \ldots, d_n$ are uniquely determined. We also assume that $z_1, \ldots, z_n$ are well separated, which means that all points $z_m q^{2(s-\Lambda_m)}$, $s = 0, \ldots, d_m - 1$, and $z_m q^{2\Lambda_m}$, $m = 1, \ldots, n$, are pairwise distinct.

Consider the second order difference equation

$$\tau(u)Q(u) = Q(q^{-2}u) \prod_{m=1}^{n} (q^{2\Lambda_m} u - z_m) + \vartheta\, Q(q^2 u) \prod_{m=1}^{n} (u - q^{2\Lambda_m} z_m)\,. \tag{6.1}$$

with respect to $Q(u)$. Here ϑ is a fixed complex number.

Let $\mathcal{S}_m = \{\, z_m q^{2(s-\Lambda_m)} \mid s = 0, \ldots, d_m \,\}$, $m = 1, \ldots, n$. Set $\mathcal{S} = \bigcup_{m=1}^{n} \mathcal{S}_m$. Let $\mathcal{F}_m = \{\, f : \mathcal{S}_m \to \mathbb{C} \,\}$ and let $\mathcal{F} = \{\, f : \mathcal{S} \to \mathbb{C} \,\}$. Let $\pi_m : \mathcal{F} \to \mathcal{F}_m$ be the canonical projection: $\pi_m \varphi = \varphi\big|_{\mathcal{S}_m}$. Set $\Lambda = \sum_{m=1}^{n} \Lambda_m$.

We consider the next problems related to difference equation (6.1).

Global problem. To determine a polynomial $\tau(u)$ such that there exists a nontrivial polynomial solution $Q(u)$ to equation (6.1) such that $Q(0) \neq 0$.

Local problem. To determine a polynomial $\tau(u)$ of degree at most n, $\tau(0) = (-1)^n (1 + q^{2\Lambda}\vartheta) \prod_{m=1}^{n} z_m$, such that there is $Q \in \mathcal{F}$, $\pi_m Q \neq 0$, $m = 1, \ldots, n$, satisfying equation (6.1) for all $u \in \mathcal{S}$.

Remark. Note that a given $Q \in \mathcal{F}$ can satisfy equation (6.1) for at most one polynomial $\tau(u)$ of degree n with the prescribed value $\tau(0)$, since the sets $\mathcal{S}_m$, $m = 1, \ldots, n$, are pairwise disjunctive.

Let $Q \in \mathcal{F}$. Say that Q is a *pseudoconstant* if all projections $\pi_m Q$ are constant functions.

(6.1) Lemma. *Let $\tau(u)$ be a solution to the local problem. Then a solution $Q \in \mathcal{F}$ to equation (6.1) is unique modulo a pseudoconstant factor.*

(6.2) Lemma. *Let $\tau(u)$ be a solution to the global problem. Then*

a) $\deg \tau \leqslant n$ *and* $\tau(0) = (-1)^n (1 + q^{2\Lambda}\vartheta) \prod_{m=1}^{n} z_m$.

b) *Let $\vartheta \neq q^{-2(s+\Lambda)}$, $s \in \mathbb{Z}_{>0}$. Then for a given $\tau(u)$, any two of the required polynomial solutions to equation (6.1) are proportional.*

Let $\tau(u)$, $Q(u)$ be a solution to the global problem. If $\tau(u)$ is also a solution to the local problem (that is if $Q\big|_{\mathcal{S}_m} \neq 0$, $m = 1, \ldots, n$), then say that $\tau(u)$ is an *admissible* global solution.

(6.3) Theorem. *Let ϑ be generic. Let $z_1, \ldots, z_n$ be well separated. Then*

a) *All solutions to the local problem are also solutions to the global problem.*

b) *The number of solutions to the local problem is equal to $\prod_{m=1}^{n} (d_m + 1)$.*

c) *If $\tau(u)$, $Q(u)$ is an admissible solution to the global problem, then $\deg Q \leqslant \sum_{m=1}^{n} d_m$.*

(6.4) Lemma. *Let $z_1, \ldots, z_n$ be well separated. Let $t_1, \ldots, t_\ell$ be an admissible offdiagonal solution to system (4.2) with $\kappa = q^{2\ell}\vartheta$. Let $\tau(u) = q^{\Lambda-\ell}\tau(u,t)$ where $\tau(u,t)$ is given by formula (4.3). Set $Q(u) = \prod_{a=1}^{\ell} (u - t_a)$. Then $\tau(u)$, $Q(u)$ is an admissible solution to the global problem.*

(6.5) Lemma. *Let M be the total number of $\mathbf{S}_\ell$-orbits of admissible solutions to system* (4.2) *for* $\ell = 0, \dots, \sum_{m=1}^{n} d_m$ *altogether. Let κ be generic. Let $z_1, \dots, z_n$ be well separated. Then* $M = \prod_{m=1}^{n} (d_m + 1)$. *Moreover, there are no admissible offdiagonal solutions to system* (4.2) *for* $\ell > \sum_{m=1}^{n} d_m$.

Remark. Since for generic κ the number of nontrivial solutions to system (4.2) does not depend on κ, the explicit dependence $\kappa = q^{2\ell}\vartheta$ is irrelevant.

Lemmas 6.4 and 6.5 give the required number of admissible solutions to the global problem. To get an estimate from above for the number of local solutions we consider a spectral problem which can be solved by separation of variables. Equation (6.1) is the equation for separated variables in this problem.

Let $\mathcal{F}_\otimes = \mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_n$. We consider $\mathcal{F}_\otimes$ as a space of functions in n variables $x_1 \in \mathcal{S}_1, \dots, x_n \in \mathcal{S}_n$. Let $y_k^\pm \in \mathrm{End}\,(\mathcal{F}_\otimes)$, $k = 1, \dots, n$, be defined as follows:

$$y_k^+ f(x_1, \dots, x_n) = f(x_1, \dots, q^{-2}x_k, \dots, x_n) \prod_{m=1}^{n} (q^{2\Lambda_m} x_k - z_m)\,,$$

$$y_k^- f(x_1, \dots, x_n) = f(x_1, \dots, q^{2}x_k, \dots, x_n) \prod_{m=1}^{n} (x_k - q^{2\Lambda_m} z_m)\,.$$

Set

$$\mathcal{T}(u) = (1 + q^{2\Lambda}\vartheta) \prod_{m=1}^{n} z_m (u/x_m - 1) + \sum_{m=1}^{n} \frac{u}{x_m} \prod_{\substack{k=1\\k\neq m}}^{n} \frac{u - x_k}{x_m - x_k} \cdot (y_m^+ + \vartheta\, y_m^-)\,.$$

(6.6) Lemma. *Coefficients of the polynomial $\mathcal{T}(u)$ generate a commutative subalgera in* $\mathrm{End}\,(\mathcal{F}_\otimes)$.

(6.7) Lemma. *Let $\tau(u)$, $Q(u)$ be a solution to the local problem. Set $Q_\otimes = \pi_1 Q \otimes \dots \otimes \pi_n Q \neq 0$. Then $\mathcal{T}(u) Q_\otimes = \tau(u) Q_\otimes$. Moreover, any eigenvector of $\mathcal{T}(u)$ has the form $Q_\otimes$ for a suitable solution $\tau(u)$, $Q(u)$ to the local problem.*

Now let q^2 be a primitive N-th root of unity. In this case we assume only that $q^{2\Lambda_m} \neq 0$, $m = 1, \dots, n$. Define integers $d_1, \dots, d_n$ and $s_1, \dots, s_n$ as follows. If $q^{4\Lambda_m} \in \{1, q^2, , \dots, q^{2N-2}\}$ then define $d_m \in \{1, \dots, N-1\}$ by $q^{4\Lambda_m} = q^{2d_m}$ and set $s_m = d_m - 1$. Otherwise set $d_m = N-1$, $s_m = N-1$. We assume that $z_1, \dots, z_n$ are well separated, which means that all points $z_m q^{2(s-\Lambda_m)}$, $s = 0, \dots, s_m$, and $z_m q^{2\Lambda_m}$, $m = 1, \dots, n$, are pairwise distinct.

Define sets $\mathcal{S}_1, \dots, \mathcal{S}_n$ as follows. If $q^{4\Lambda_m} \in \{1, q^2, , \dots, q^{2N-4}\}$ then define $\mathcal{S}_m = \{ z_m q^{2(s-\Lambda_m)} \mid s = 0, \dots, d_m \}$ similar to the case of generic q. Otherwise define $\mathcal{S}_m = \{ \zeta_m q^{2s} \mid s = 0, \dots, N-1 \}$ with arbitrary $\zeta_m \in \mathbb{C}_\circ$. If $\zeta_m \neq z_m q^{\pm 2\Lambda_m}$ then say that $\mathcal{S}_m$ is *cyclic*. We assume that sets $\mathcal{S}_1, \dots, \mathcal{S}_n$ are pairwise disjunctive. For instance, this is the case provided all of them are not cyclic.

(6.8) Lemma. *Let all sets $\mathcal{S}_1, \dots, \mathcal{S}_n$ be not cyclic or ϑ be generic. Let $\tau(u)$ be a solution to the local problem. Then a solution $Q \in \mathcal{F}$ to equation* (6.1) *is unique modulo a pseudoconstant factor.*

For any polynomial $P(u)$ say that $P(u^N)$ is a quasiconstant.

(6.9) Lemma. *Let $\tau(u)$ be a solution to the global problem. Then*

a) $\deg \tau \leqslant n$ *and* $\tau(0) = (-1)^n (1 + q^{2\Lambda}\vartheta) \prod_{m=1}^{n} z_m$.

b) *Let $\vartheta \notin \{ q^{-2(s+\Lambda)} \mid s = 1, \dots, N-1 \}$. Then for a given $\tau(u)$, the required polynomial solution to equation* (6.1) *is unique modulo a quasiconstant factor.*

Let $\tau(u)$, $Q(u)$ be a solution to the global problem and $Q(u)$ has the smallest possible degree. If $\tau(u)$ is also a solution to the local problem, then say that $\tau(u)$ is an *admissible* global solution.

(6.10) Theorem. *Let q be a root of unity. Let ϑ be generic. Let $z_1, \dots, z_n$ be well separated. Let $\mathcal{S}_1, \dots, \mathcal{S}_n$ be pairwise disjunctive. Then all claims of Theorem 6.3 hold.*

The proof is completely similar to the case of generic q, because all Lemmas 6.4 – 6.7 remain valid. A slight change is necessary in the proof of Lemma 6.4 if some $\mathcal{S}_m$ is cyclic. Namely, in this case $Q\big|_{\mathcal{S}_m} \neq 0$ because polynomials $Q(u)$ and $Q(q^2 u)$ have no common zeros. The last claim itself follows from the fact that $t_1, \dots, t_\ell$ is an admissible solution to system (4.2).

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